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Shear layers in converging flow of fluid of non-uniform density and viscosity

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We present an asymptotic study of steady two-dimensional radial flow between converging plane walls (Jeffery–Hamel flow) when the viscosity μ and density ρ vary with the angular coordinate θ. Two representative situations are considered, the first being a two-layer system (in which μ and ρ are uniform except for discontinuities at an interface $\theta = \theta_{\mathrm{I}}$), and the other involving a fluid for which μ and ρ vary continuously with θ. The flow is analysed in the asymptotic limit when a parameter c related to the wall pressure gradient is large; this corresponds to converging flow at large Reynolds number. Solutions are derived for the boundary layers at the walls and for the shear layer at the interface; the results are shown to agree well with some exact (numerical) profiles.

The solutions obtained are not unique, though for given c they represent the 'simplest' type of profile, and the one that seems most likely to be stable. We demonstrate the non-uniqueness by deriving in §3 an alternative solution for the interfacial shear layer. This solution, however, can exist for only restricted ranges of values of the density and viscosity ratios, and involves an outgoing jet, suggesting that it is likely to be unstable.

1. Introduction

Flow systems involving fluids that are stratified in viscosity and density occur commonly, in industrial contexts, for example. Unfortunately such flows tend to be very complex, and detailed descriptions are rare. One of the more tractable problems in this general class was discussed recently by Hooper, Duffy & Moffatt (1982, to be referred to as I), who considered two-dimensional viscous stratified flow between plane converging or diverging walls – a generalization of the conventional Jeffery–Hamel flow (Rosenhead 1940; Fraenkel 1962). This problem is of relevance to the study of extrusion processes, though in I it developed from a study of stratified flow in ducts of slowly varying cross-section.

It was shown in I that Jeffery–Hamel (JH) flow can provide an exact solution of the Navier–Stokes equations even when the density ρ and viscosity μ of the fluid vary with the angular coordinate θ. The JH similarity assumption reduces the Navier–Stokes equations to a second-order nonlinear ordinary differential equation whose coefficients depend upon $\mu(\theta)$ and $\rho(\theta)$. In particular, μ and ρ may vary discontinuously with θ; that is, steady radial flow is possible even for a 'layered' fluid (with μ and ρ differing from one layer to another) provided that the interface between any two layers coincides with a coordinate surface θ = constant. At such an interface, the velocity and shear stress are continuous, as also is the normal stress, since the interface remains plane, and interfacial tension plays no part.

In I, two configurations were analysed in some detail, namely the single-fluid case, and a two-layer case with a jump in viscosity across $\theta = 0$, the density being uniform. In this paper we consider converging flow, at high Reynolds number, when there is variation of both μ and ρ. Boundary-layer solutions for various situations are presented, these solutions being in the form of asymptotic expansions in a large parameter c related to the pressure gradient at the wall.

First, in §2, a two-layer flow is considered in which both the viscosity ratio λ and the density ratio σ^2 of the two phases are arbitrary. The form of the boundary layer near each wall and of the shear layer at the interface is derived, and these asymptotic results are compared with some exact profiles obtained numerically. The type of solution given in §2 is by no means unique, and in §3 an alternative profile for the region near the interface is obtained. However, this second type of solution (which, for given σ, can exist for only a restricted range of values of λ), involves an outgoing jet, and is likely to be unstable.

For completeness, we obtain in §4 the boundary-layer expansion for the case when μ and ρ vary continuously with θ; if ρ is not constant, then the outer (inviscid) flow is rotational. The analysis here requires an assumption that buoyancy effects are negligible, with variation of density important only insofar as the advective acceleration of particles is concerned (so that, in particular, ρ may be constant).

2. General formulation, and two-layer flow

We consider plane radial flow of incompressible viscous fluid between converging walls, referring the description to a cylindrical polar coordinate system (r, θ, z), with the bounding walls at $\theta = \pm\alpha$ (see figure 1*a*). We allow the viscosity μ and density ρ of the fluid to vary with θ, and define

$$\bar{\mu} = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} \mu(\theta)\, d\theta, \quad \bar{\rho} = \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} \rho(\theta)\, d\theta, \quad \tilde{\mu} = \frac{\mu}{\bar{\mu}}, \quad \tilde{\rho} = \frac{\rho}{\bar{\rho}}, \tag{2.1a}$$

so that

$$\int_{-\alpha}^{\alpha} \tilde{\mu}(\theta)\, d\theta = \int_{-\alpha}^{\alpha} \tilde{\rho}(\theta)\, d\theta = 2\alpha. \tag{2.1b}$$

Then, with a radial velocity

$$u(r,\theta) = \frac{2\bar{\mu}}{\bar{\rho}} \frac{f(\theta)}{r} \tag{2.2}$$

the function $f(\theta)$ satisfies (see I)

$$\tilde{\mu} f'' + \tilde{\mu}' f' + 4\tilde{\mu} f + 2\tilde{\rho} f^2 = c, \tag{2.3}$$

and the pressure is

$$p = p_0 + \frac{\bar{\mu}^2}{\bar{\rho} r^2}(4\tilde{\mu} f - c), \tag{2.4}$$

where c is a constant related to the wall pressure gradient. The no-slip boundary condition at the walls $\theta = \pm\alpha$ means that

$$f(\pm\alpha) = 0, \tag{2.5}$$

and the Reynolds number based on the overall flux is

$$R = \int_{-\alpha}^{\alpha} f(\theta)\, d\theta. \tag{2.6}$$

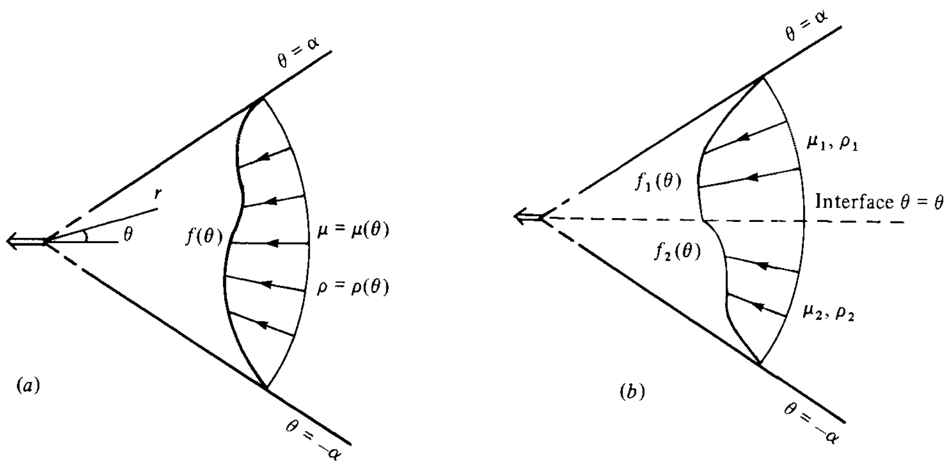

FIGURE 1. Jeffery–Hamel flow with angular variation of density and viscosity: (a) continuous variation; (b) the two-layer flow studied in §§2–3.

We consider first a two-layer flow (figure 1b) in which μ and ρ are uniform in each phase but are discontinuous across $\theta = \theta_{\mathrm{I}}$, with

$$\tilde{\mu}(\theta) = \begin{cases}\tilde{\mu}_1, \\ \tilde{\mu}_2,\end{cases} \quad \tilde{\rho}(\theta) = \begin{cases}\tilde{\rho}_1 & (\theta_{\mathrm{I}} < \theta < \alpha), \\ \tilde{\rho}_2 & (-\alpha < \theta < \theta_{\mathrm{I}}),\end{cases} \tag{2.7}$$

$$\bar{\mu} = \frac{1}{2}\left[(\mu_1+\mu_2)+\frac{\theta_{\mathrm{I}}}{\alpha}(\mu_2-\mu_1)\right], \quad \bar{\rho} = \frac{1}{2}\left[(\rho_1+\rho_2)+\frac{\theta_{\mathrm{I}}}{\alpha}(\rho_2-\rho_1)\right].$$

Writing
$$f(\theta) = \begin{cases} f_1(\theta) & (\theta_{\mathrm{I}} < \theta < \alpha), \\ f_2(\theta) & (-\alpha < \theta < \theta_{\mathrm{I}}),\end{cases} \tag{2.8}$$

(2.3) becomes
$$\tilde{\mu}_1(f_1''+4f_1)+2\tilde{\rho}_1 f_1^2 = c = \tilde{\mu}_2(f_2''+4f_2)+2\tilde{\rho}_2 f_2^2, \tag{2.9}$$

and continuity of velocity and shear stress across the interface requires that

$$[f]_-^+ = 0, \quad [\tilde{\mu} f']_-^+ = 0 \quad \text{on} \quad \theta = \theta_{\mathrm{I}}. \tag{2.10}$$

Continuity of normal stress at $\theta = \theta_{\mathrm{I}}$ is achieved by taking c to be the same for the two fluids. We shall take the interface to be horizontal; gravity can then be accounted for simply by modifying the pressure (and of course, with the heavier fluid below the lighter, gravity will tend to stabilize the flow).

The constant c in (2.9) will be taken to be large; for the simplest type of velocity profile (see I), a large value of c corresponds to converging flow with boundary layers at the walls, R being large and negative. These boundary layers have the asymptotic structure†

$$\left.\begin{aligned} f &\sim \sum_{n\geqslant 0} a_{2n}(\tilde{\mu},\tilde{\rho})\, K^{2-2n} F_{2n}(\eta), \\ \text{where} \qquad \eta &= \tilde{\rho}^{\frac{1}{4}}\tilde{\mu}^{-\frac{1}{2}} K(\alpha\pm\theta), \quad K \equiv (\tfrac{1}{2}c)^{\frac{1}{4}} \gg 1. \end{aligned}\right\} \tag{2.11}$$

† In this two-layer problem, one could obtain an exact solution in terms of elliptic functions, and then from this derive the asymptotic results (2.11) and (2.25) using standard formulae.

The constants a_{2n} here are introduced simply in order to avoid carrying multiplicative factors in the F_{2n}; also the symbols $(f, \tilde{\mu}, \tilde{\rho}, \eta, \pm)$ are intended to represent $(f_1, \tilde{\mu}_1, \tilde{\rho}_1, \eta_1, -)$ or $(f_2, \tilde{\mu}_2, \tilde{\rho}_2, \eta_2, +)$, as appropriate. Outside the boundary layers, the derivative terms in (2.9) may be neglected, so that in each fluid

$$f \sim -\frac{\tilde{\mu}}{\tilde{\rho}}\left\{1+\left[1+\frac{c\tilde{\rho}}{2\tilde{\mu}^2}\right]^{\frac{1}{2}}\right\} \sim -\frac{K^2}{\tilde{\rho}^{\frac{1}{2}}}-\frac{\tilde{\mu}}{\tilde{\rho}}-\frac{\tilde{\mu}^2}{2\tilde{\rho}^{\frac{3}{2}}K^2}, \tag{2.12a}$$

which we may write as

$$f \sim \sum_{n \geqslant 0} b_{2n}(\tilde{\mu}, \tilde{\rho})\, K^{2-2n}, \tag{2.12b}$$

where the b_{2n} are known constants. However, for $\eta \rightarrow \infty$ (2.11) becomes

$$f \sim \sum_{n \geqslant 0} a_{2n}(\tilde{\mu}, \tilde{\rho})\, K^{2-2n} F_{2n}(\infty), \tag{2.13}$$

and, comparing this with (2.12), we see that we may impose the following boundary conditions on the F_{2n} at $\eta = \infty$:

$$\left.\begin{array}{lll} F_{2n}(\infty) = -1, & a_{2n} = -b_{2n} & (b_{2n} \neq 0), \\ F_{2n}(\infty) = 0, & a_{2n} \text{ arbitrary but non-zero} & (b_{2n} = 0). \end{array}\right\} \tag{2.14}$$

However, in practice it is usually sufficient (and simpler) to impose instead $F'_{2n}(\infty) = 0$. Clearly

$$a_0 = \tilde{\rho}^{-\frac{1}{2}}, \quad a_2 = \tilde{\mu}\tilde{\rho}^{-1}, \quad a_4 = \tfrac{1}{2}\tilde{\mu}^2\tilde{\rho}^{-\frac{3}{2}}. \tag{2.15}$$

Substitution of (2.11) into (2.9) leads to the following set of equations for the F_{2n}:

$$F''_0 = 2(1-F_0^2), \tag{2.16}$$

$$\mathscr{L}(F_{2n}) \equiv F''_{2n}+4F_0 F_{2n} = G_{2n}(F_0, F_2, \ldots, F_{2n-2}) \quad (n > 0); \tag{2.17}$$

these are to be integrated subject to the boundary conditions

$$F_{2n}(0) = 0, \quad F'_{2n}(\infty) = 0. \tag{2.18}$$

Recognizing that $F'_0(\eta)$ is a complementary function for (2.17), one can easily construct the solution

$$F_{2n}(\eta) = F'_0(\eta)\int_0^{\eta}\left\{[F'_0]^{-2}\int_{\infty}^{\xi} G_{2n}F'_0\, d\zeta\right\}d\xi \quad (n > 0), \tag{2.19}$$

and then with

$$G_2 = -4F_0, \quad G_4 = 4-4(1+F_2)^2, \tag{2.20}$$

the first few F_{2n} are found to be† (cf. §5 of I)

$$F_0(\eta) = 2-3\tanh^2(\eta+\beta), \tag{2.21}$$

$$F_2(\eta) = 3(\tfrac{3}{2})^{\frac{1}{2}}\tanh(\eta+\beta)\,\mathrm{sech}^2(\eta+\beta)-1, \tag{2.22}$$

$$F_4(\eta) = \tfrac{3}{8}\mathrm{sech}^2(\eta+\beta)\,[18\tanh^2(\eta+\beta)-(8\eta+3\surd 6)\tanh(\eta+\beta)+2]-1, \tag{2.23}$$

where $\beta = \mathrm{artanh}\,(\tfrac{2}{3})^{\frac{1}{2}} \approx 1{\cdot}146$.

Away from the walls, (2.12) gives, in the two fluids,

$$f_1(\theta) \sim -\tilde{\rho}_1^{-\frac{1}{2}}K^2-\tilde{\rho}_1^{-1}\tilde{\mu}_1, \quad f_2(\theta) \sim -\tilde{\rho}_2^{-\frac{1}{2}}K^2-\tilde{\rho}_2^{-1}\tilde{\mu}_2. \tag{2.24}$$

† A boundary layer with an outgoing wall jet is also possible (see e.g. Rosenhead 1963, p. 236); here we concentrate on the simpler profile, involving only inflow. Note that the notation for the Fs differs slightly from that in I.

Clearly there must be a shear layer near the interface $\theta = \theta_I$ where viscous effects are important in the transition between the two 'outer' flows (2.24). This shear layer will have the asymptotic structure

$$f(\theta) = \begin{cases} \tilde{\rho}_1^{-\frac{1}{2}} K^2 f_{10}(\eta_1) + \tilde{\rho}_1^{-1} \tilde{\mu}_1 f_{12}(\eta_1) + O(K^{-2}), & \eta_1 = \tilde{\rho}_1^{\frac{1}{4}} \tilde{\mu}_1^{-\frac{1}{2}} K(\theta - \theta_I) \quad (\theta > \theta_I), \\ \tilde{\rho}_2^{-\frac{1}{2}} K^2 f_{20}(\eta_2) + \tilde{\rho}_2^{-1} \tilde{\mu}_2 f_{22}(\eta_2) + O(K^{-2}), & \eta_2 = \tilde{\rho}_2^{\frac{1}{4}} \tilde{\mu}_2^{-\frac{1}{2}} K(\theta - \theta_I) \quad (\theta < \theta_I) \end{cases} \tag{2.25}$$

Substituting into (2.9) and (2.10) we have at leading order in K

$$f''_{i0} = 2(1 - f_{i0}^2) \quad (i = 1, 2), \tag{2.26}$$

with boundary conditions

$$f_{10}(0) = \sigma f_{20}(0), \quad \lambda^{\frac{1}{2}} f'_{10}(0) = \sigma^{\frac{1}{2}} f'_{20}(0), \quad f'_{10}(\infty) = f'_{20}(-\infty) = 0, \tag{2.27}$$

where

$$\sigma = \left(\frac{\tilde{\rho}_1}{\tilde{\rho}_2}\right)^{\frac{1}{2}}, \quad \lambda = \frac{\tilde{\mu}_1}{\tilde{\mu}_2}. \tag{2.28}$$

Without loss of generality we may restrict σ by $0 < \sigma < 1$, while allowing λ to have any positive value (the case $\sigma = 1$ was treated in I). Then the appropriate solutions to (2.26), (2.27) are†

$$f_{10}(\eta_1) = 2 - 3\tanh^2(\eta_1 + \beta_1), \quad f_{20}(\eta_2) = 2 - 3\coth^2(\eta_2 + \beta_2), \tag{2.29a, b}$$

with the boundary conditions at $\theta = \theta_I$ requiring that

$$2 - 3\tanh^2 \beta_1 = \sigma(2 - 3\coth^2 \beta_2), \tag{2.30a}$$

$$\lambda^{\frac{1}{2}} \tanh \beta_1 \operatorname{sech}^2 \beta_1 = -\sigma^{\frac{1}{2}} \coth \beta_2 \operatorname{cosech}^2 \beta_2. \tag{2.30b}$$

Now, for f_{20} to remain finite in $\eta_2 < 0$, β_2 must be negative, and so from (2.30*b*) β_1 must be positive; then (2.30*a*) shows that in fact $\beta_1 > \beta$.‡ Thus we may write

$$\beta_1 = \operatorname{artanh}\left(\frac{\sigma b + 2}{3}\right)^{\frac{1}{2}} > \beta, \quad \beta_2 = -\operatorname{arcoth}\left(\frac{b+2}{3}\right)^{\frac{1}{2}} < 0 \quad (1 < b < \sigma^{-1}), \tag{2.31}$$

and β_1 and β_2 are given uniquely in terms of the parameter b which, from (2.30*b*), must satisfy

$$(1 - \lambda\sigma^2) b^3 - 3(1 - \lambda) b - (2/\sigma)(\lambda - \sigma) = 0, \tag{2.32}$$

a cubic equation that has a unique real root in the range $1 < b < \sigma^{-1}$. Clearly if $\lambda\sigma^2 = 1$ (so that $\mu_1 \rho_1 = \mu_2 \rho_2$) then

$$b = \frac{2(1 + \sigma + \sigma^2)}{3\sigma(1 + \sigma)};$$

† It might have been expected that f_{10} and f_{20} would *both* be of the form (2.29*a*), involving tanh rather than coth. However the boundary conditions at $\theta = \theta_I$ could then be satisfied only for restricted ranges of values of σ and λ (see §3); in particular, there could be no solution with $\lambda = 1$ (unless also $\sigma = 1$, in which case $\beta_1 = -\beta_2 = \infty$ and $f_{10} = f_{20} = -1$). Note that, if we had chosen to make $\sigma > 1$, then tanh and coth in (2.29) would be interchanged.

‡ This means, incidentally, that $f_{10} < 0$ in $\eta_1 > 0$; and, since also $f_{20} < 0$ in $\eta_2 < 0$, the flow is purely inward, with no outgoing 'interior jet' (cf. §3).

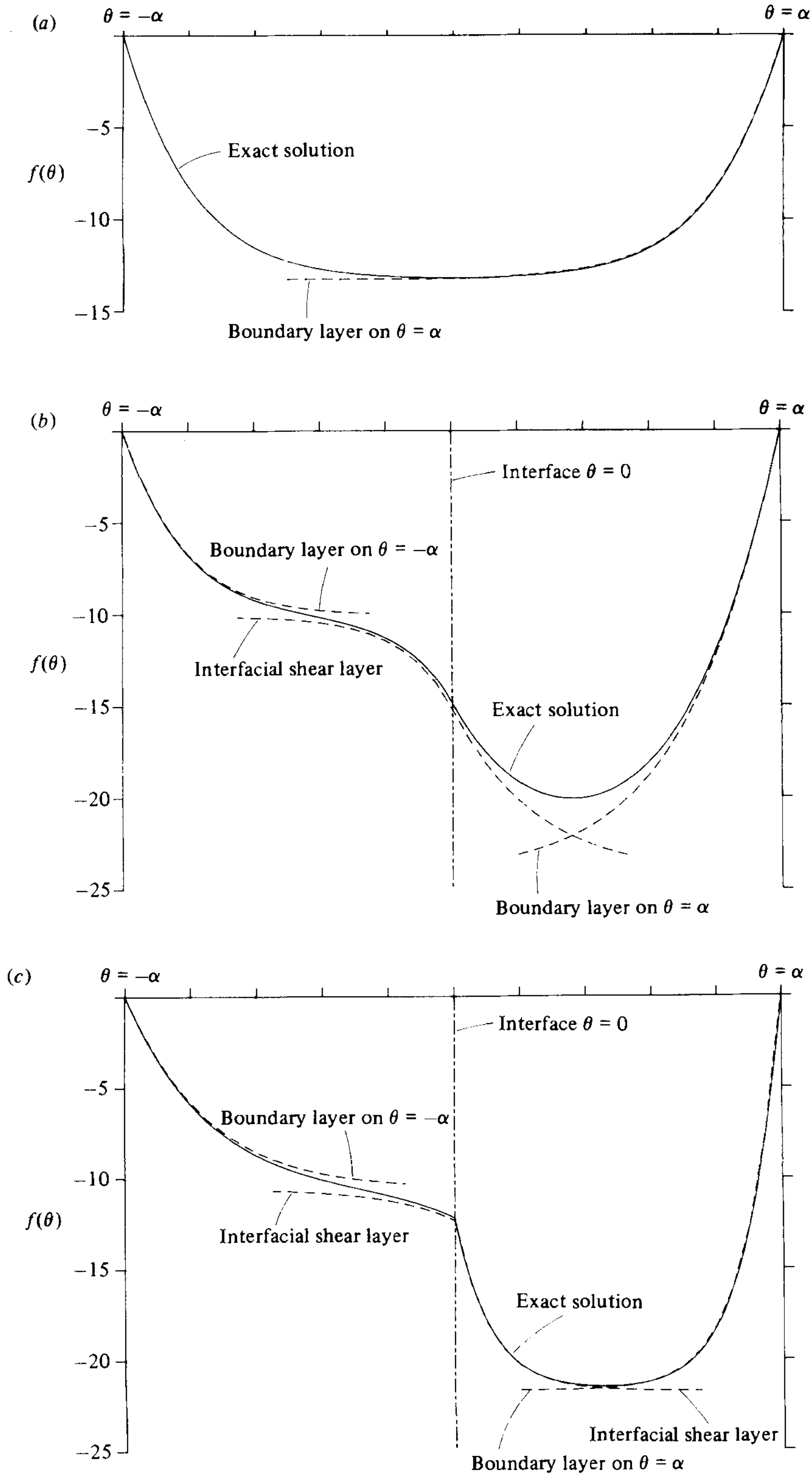

FIGURE 2 (a, b, c). For caption see facing page.

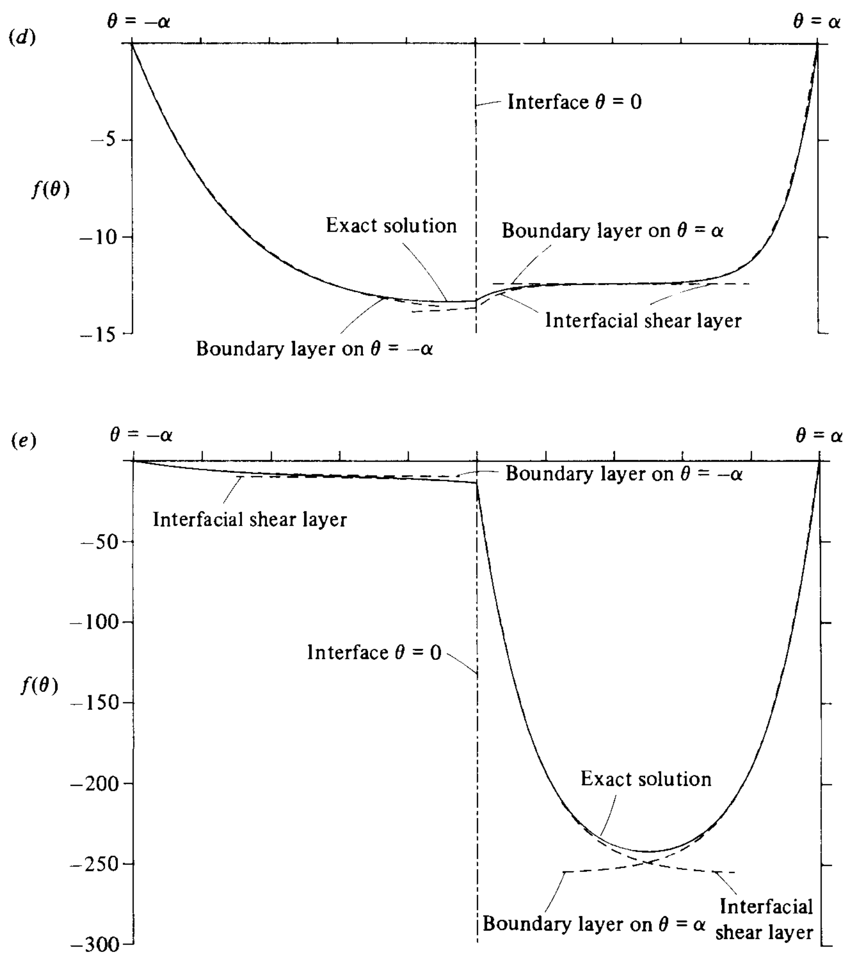

FIGURE 2. Comparison of some exact solutions with the two-term asymptotic results (2.11) (for the boundary layers at $\theta = \pm\alpha$) and (2.25) (for the shear layer at $\theta = \theta_I$). The curves are drawn for the case $c = 300$ ($K \approx 3{\cdot}5$), $\alpha = \frac{1}{4}\pi$ and $\theta_I = 0$, with (*a*) $\lambda = 1$, $\sigma^2 = 1$; (*b*) $\lambda = 1$, $\sigma^2 = 0{\cdot}2$; (*c*) $\lambda = 0{\cdot}1$, $\sigma^2 = 0{\cdot}2$; (*d*) $\lambda = 0{\cdot}1$, $\sigma^2 = 1$; (*e*) $\lambda = 1{\cdot}35 \times 10^{-2}$, $\sigma^2 = 1{\cdot}25 \times 10^{-3}$.

if $\lambda = \sigma$ (so that $\mu_1^{\frac{1}{2}}\rho_1^{-\frac{1}{4}} = \mu_2^{\frac{1}{2}}\rho_2^{-\frac{1}{4}}$, and the boundary layers in the two fluids are of the same thickness) then

$$b = \left[\frac{3}{1+\sigma+\sigma^2}\right]^{\frac{1}{2}};$$

and if $\lambda = 1$ ($\mu_1 = \mu_2$) then

$$b = \left[\frac{2}{\sigma(1+\sigma)}\right]^{\frac{1}{3}}.$$

Also, if $\sigma \to 1$, then $b \to 1$ and the solution (6.4) of I is recovered.

At next order we have

$$(f_{i2}+1)''+4f_{i0}(f_{i2}+1) = 0 \quad (i = 1,2), \tag{2.33a}$$

with boundary conditions

$$\lambda f_{12}(0) = \sigma^2 f_{22}(0), \quad \lambda^{\frac{3}{2}} f'_{12}(0) = \sigma^{\frac{3}{2}} f'_{22}(0), \quad f'_{12}(\infty) = f'_{22}(-\infty) = 0. \tag{2.33b}$$

The solution is

$$\left.\begin{aligned} f_{12}(\eta_1) &= A_1 \tanh(\eta_1+\beta_1)\,\mathrm{sech}^2(\eta_1+\beta_1)-1, \\ f_{22}(\eta_2) &= A_2 \coth(\eta_2+\beta_2)\,\mathrm{cosech}^2(\eta_2+\beta_2)-1, \end{aligned}\right\} \tag{2.34}$$

where

$$\left.\begin{aligned} A_1 &= \frac{\sigma^{\frac{3}{2}}A_0(b+1)}{1-\sigma b}, \quad A_2 = \frac{\lambda^{\frac{3}{2}}A_0(1+\sigma b)}{b-1}, \\ A_0 &= \left(\frac{27}{\sigma}\right)^{\frac{1}{2}}(\lambda-\sigma^2)\,[\sigma\lambda(2+\sigma b)^{\frac{1}{2}}(b+1)+(\sigma\lambda)^{\frac{3}{2}}(2+b)^{\frac{1}{2}}(1+\sigma b)]^{-1} \end{aligned}\right\} \tag{2.35}$$

(so that, should the fluids have the same kinematical viscosity μ/ρ, then $A_1 = A_2 = 0$ and the $O(1)$ terms in (2.25) would have simply $f_{i2} = -1$; however, higher-order terms would not be constants). To this order of approximation the relation (2.6) becomes

$$\begin{aligned} R \sim {} & -\alpha K^2(\tilde{\rho}_1^{-\frac{1}{2}}+\tilde{\rho}_2^{-\frac{1}{2}})+3K\{\tilde{\mu}_1^{\frac{1}{2}}\tilde{\rho}_1^{-\frac{3}{4}}[2-(\tfrac{2}{3})^{\frac{1}{2}}-\tanh\beta_1]+\tilde{\mu}_2^{\frac{1}{2}}\tilde{\rho}_2^{-\frac{3}{4}}[2-(\tfrac{2}{3})^{\frac{1}{2}}+\coth\beta_2]\} \\ & -\alpha(\tilde{\mu}_1\tilde{\rho}_1^{-1}+\tilde{\mu}_2\tilde{\rho}_2^{-1})+\frac{1}{2K}\{\tilde{\mu}_1^{\frac{3}{2}}\tilde{\rho}_1^{-\frac{5}{4}}[(\tfrac{3}{2})^{\frac{1}{2}}+A_1\,\mathrm{sech}^2\beta_1]+\tilde{\mu}_2^{\frac{3}{2}}\tilde{\rho}_2^{-\frac{5}{4}}[(\tfrac{3}{2})^{\frac{1}{2}}-A_2\,\mathrm{cosech}^2\beta_2]\}. \end{aligned} \tag{2.36}$$

Figure 2 compares the above asymptotic results (drawn as dashed lines) with the corresponding exact computed profiles (drawn as full lines), for the case $\alpha = \frac{1}{4}\pi$, $\theta_{\mathrm{I}} = 0$ and $c = 300$ ($K \approx 3{\cdot}5$). Figure 2(*a*) shows the uniform-fluid case $\sigma^2 = \lambda = 1$, together with the asymptotic expansion (2.11) for the boundary layer on $\theta = \alpha$. Figures 2(*b–e*) show the modifications to the profile when σ^2 and λ are changed, figure 2(*e*) corresponding to an air–water system ($\lambda = 1{\cdot}35\times10^{-2}$, $\sigma^2 \approx 1{\cdot}25\times10^{-3}$, with both fluids regarded as incompressible). The values of the Reynolds number predicted by (2.36) for these cases are respectively $R = -16{\cdot}61$, $-19{\cdot}32$, $-20{\cdot}99$, $-16{\cdot}95$, $-150{\cdot}03$, agreeing with the computed values to within $\frac{3}{4}$% at worst.

As one simple application of the results, we can consider briefly the situation sketched in figure 14(*a*) of I – a two-layer flow through a converging duct into a parallel-sided channel. If the interface is at $\theta = 0$ in the converging section, then for sufficiently large c the fluxes Q_1 and Q_2 of the two fluids are approximately

$$\{Q_1, Q_2\} = -2\alpha(\tfrac{1}{2}c)^{\frac{1}{2}}\frac{\bar{\mu}}{\bar{\rho}}\{\tilde{\rho}_1^{-\frac{1}{2}}, \tilde{\rho}_2^{-\frac{1}{2}}\},$$

so that $q \equiv Q_1/Q_2 = \sigma^{-1}$. Downstream in the straight section the depths of the two layers are in the ratio $(1-H)/H$, where H is the solution of (I 2.16) with $q = \sigma^{-1}$. For example, if $\mu_1 = \mu_2$ ($\lambda = 1$), then

$$H = \tfrac{1}{2}-\sin\left[\tfrac{1}{3}\arcsin\frac{1-\sigma}{1+\sigma}\right] \quad (0 < H < 1).$$

On the other hand, for an air–water system $H \approx 0{\cdot}41$.

3. Solutions involving interior jets

The leading-order equation (2.26), as well as having the solution (2.29), also has a solution

$$f_{10}(\eta_1) = 2-3\tanh^2(\eta_1+\beta_1), \quad f_{20}(\eta_2) = 2-3\tanh^2(\eta_2+\beta_2), \tag{3.1}$$

provided that there exist real constants β_1 and β_2 such that

$$2-3\tanh^2\beta_1 = \sigma(2-3\tanh^2\beta_2), \quad \lambda^{\frac{1}{2}}\tanh\beta_1\,\mathrm{sech}^2\beta_1 = \sigma^{\frac{1}{2}}\tanh\beta_2\,\mathrm{sech}^2\beta_2. \tag{3.2}$$

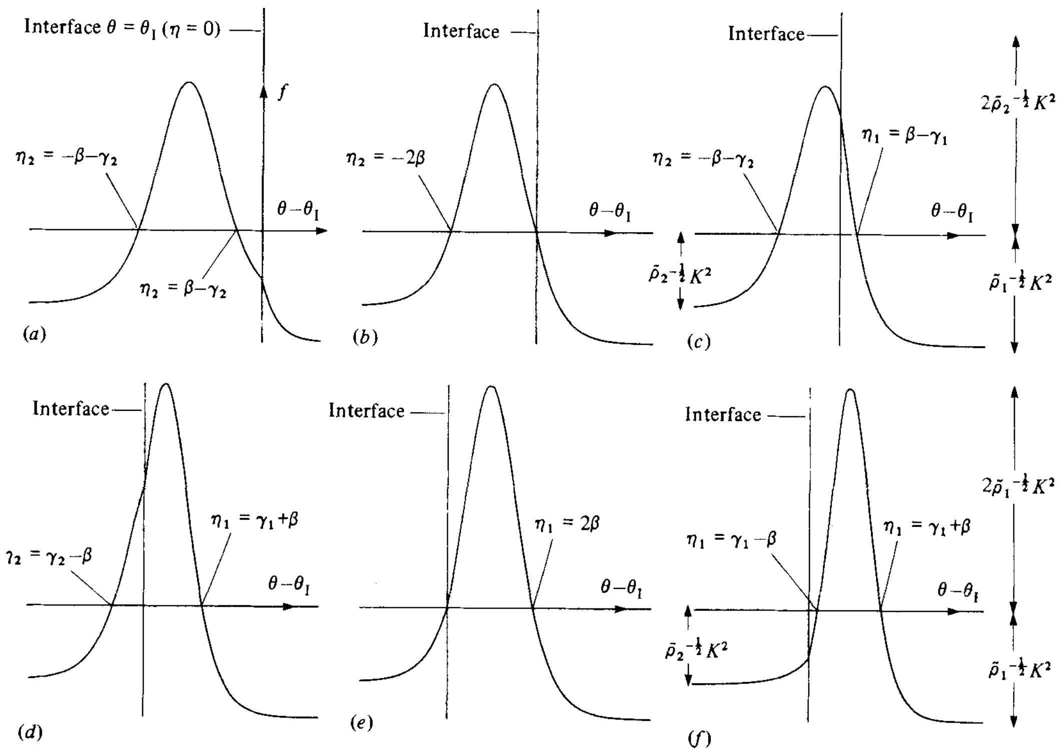

FIGURE 3. Sketch of possible jet-like profiles near the interface $\theta = \theta_I$, as given by (3.1). Fluid 1 is in $\theta > \theta_I$, and fluid 2 in $\theta < \theta_I$. A positive value of f corresponds to outflow; away from the interface the flow is inward, with f tending to the values $-\tilde{\rho}_1^{-\frac{1}{2}}K^2$ and $-\tilde{\rho}_2^{-\frac{1}{2}}K^2$. The interface is moving inwards in (*a*) and (*f*), outwards in (*c*) and (*d*), and is stationary in (*b*) and (*e*). The characteristics of the six profiles here are summarized in table 1.

Defining
$$\gamma_1 = \operatorname{artanh}\left(\frac{2+\sigma b}{3}\right)^{\frac{1}{2}} > 0, \quad \gamma_2 = \operatorname{artanh}\left(\frac{2+b}{3}\right)^{\frac{1}{2}} > 0, \tag{3.3}$$
with
$$-2 < b < 1, \tag{3.4}$$
the possible solutions of (3.2) are
$$\left.\begin{array}{lll} \text{(i)} & \beta_1 = \gamma_1, & \beta_2 = \gamma_2, \\ \text{(ii)} & \beta_1 = -\gamma_1, & \beta_2 = -\gamma_2, \end{array}\right\} \tag{3.5}$$
provided that b is a real root of the cubic equation (2.32) satisfying (3.4). Such a root exists if and only if
$$\lambda \leqslant \Lambda(\sigma), \quad \Lambda(\sigma) \equiv \frac{\sigma(2+\sigma)}{1+2\sigma} \quad (\sigma < 1), \tag{3.6}$$
and, when this condition holds, (2.32) in fact has *two* relevant roots b, one of which is negative, and the other either positive, zero or negative according as $\lambda < \sigma$, $\lambda = \sigma$ or $\lambda > \sigma$; if $\lambda = \Lambda(\sigma)$, these two roots coincide, at $b = -(1+\sigma)^{-1}$. In the range $0 < \sigma < 1$, the function $\Lambda(\sigma)$ increases monotonically from 0 to 1, and satisfies $\sigma < \Lambda(\sigma) < \sigma^{\frac{1}{2}}$.

So, (3.1) can be the leading-order solution only if (3.6) is true; and then, for given $\sigma < 1$ and $\lambda \leqslant \Lambda$, (3.1) actually represents several different profiles, since in general there are two possible values of b, for each of which there are two possibilities (3.5).

	Sign of β_1 and β_2	Zeros of velocity		Maximum of velocity	Corresponding diagram in figure 3
$b > 0\,(\gamma_2 > \gamma_1 > \beta > 0)$	$+$	$\eta_2 = \beta - \gamma_2$,	$\eta_2 = -\beta - \gamma_2$	$\eta_2 = -\gamma_2$	(*a*)
	$-$	$\eta_1 = \gamma_1 + \beta$,	$\eta_1 = \gamma_1 - \beta$	$\eta_1 = \gamma_1$	(*f*)
$b = 0\,(\gamma_1 = \gamma_2 = \beta)$	$+$	$\eta_1 = \eta_2 = 0$,	$\eta_2 = -2\beta$	$\eta_2 = -\beta$	(*b*)
	$-$	$\eta_1 = \eta_2 = 0$,	$\eta_1 = 2\beta$	$\eta_1 = \beta$	(*e*)
$b < 0\,(\beta > \gamma_1 > \gamma_2 > 0)$	$+$	$\eta_1 = \beta - \gamma_1$,	$\eta_2 = -\beta - \gamma_2$	$\eta_2 = -\gamma_2$	(*c*)
	$-$	$\eta_1 = \beta + \gamma_1$,	$\eta_2 = \gamma_2 - \beta$	$\eta_1 = \gamma_1$	(*d*)

TABLE 1. Characteristics of the profiles (3.1); the qualitative appearance of the profiles depends on the sign of $b(\lambda, \sigma)$ (the solution of (2.32) satisfying (3.4)) and on the choice of sign of β_1 and β_2.

In each case the velocity profile involves a single outgoing 'interior jet', the solution (3.1) having two zeros between which f_0 is positive (with a maximum value $f_0 = 2$). When $b > 0$, the jet is wholly within one or other of the fluids, but when $b < 0$ it straddles the interface (which therefore moves with a positive, outward velocity); when $b = 0$ $(\lambda = \sigma)$ the interface is stationary. Figure 3 shows sketches of these various possibilities, and some details are summarized in table 1.

The asymptotic expansion may be continued as in (2.25). At $O(K^0)$ the f_{i2}, again satisfying (2.33), are

$$\left.\begin{aligned} f_{12}(\eta_1) &= B_1 \tanh(\eta_1+\beta_1)\operatorname{sech}^2(\eta_1+\beta_1) - 1,\\ f_{22}(\eta_2) &= B_2 \tanh(\eta_2+\beta_2)\operatorname{sech}^2(\eta_2+\beta_2) - 1,\end{aligned}\right\} \tag{3.7}$$

where

$$B_1 = \frac{\sigma^{\frac{3}{2}} B_0(b+1)}{1-\sigma b}, \quad B_2 = -\frac{\lambda^{\frac{3}{2}} B_0(1+\sigma b)}{b-1}, \tag{3.8}$$

with

$$B_0 = \pm\left(\frac{27}{\sigma}\right)^{\frac{1}{2}} (\lambda-\sigma^2)\,[\lambda\sigma(1+b)\,(2+\sigma b)^{\frac{1}{2}} - (\lambda\sigma)^{\frac{3}{2}}\,(1+\sigma b)\,(2+b)^{\frac{1}{2}}]^{-1};$$

here the $\pm$ sign corresponds to choices (i) and (ii) respectively in (3.5).

Perhaps it should be said that the shear-layer solutions obtained earlier (§2) seem more 'fundamental' than those considered here, in that the earlier ones can (in principle) occur for any values of λ and σ, and also are simpler, having no stationary points and no positive jets. We may also mention that there can be no shear-layer solution of the form

$$f_{10}(\eta_1) = 2 - 3\coth^2(\eta_1+\beta_1), \quad f_{20}(\eta_2) = 2 - 3\coth^2(\eta_2+\beta_2),$$

since this would require that $\beta_1 > 0$ and $\beta_2 < 0$ (for the velocity to be finite everywhere), whereas the boundary condition (2.27*b*) would require β_1 and β_2 to have the same sign.

4. Continuous stratification

In the two-layer problems considered above, the outer solution (2.12) is a constant in each fluid, and so the corresponding velocity (2.2) represents a potential flow – vorticity is generated only at the (plane) boundaries and interface, and is then confined to regions near these special planes. If, however, the density varies continuously with θ (and $d\tilde{\rho}/d\theta$ is non-zero) then constant-pressure surfaces do not coincide with constant-density surfaces, and vorticity is generated throughout the inviscid stream. Therefore at leading order the outer flow – the solution of the inviscid JH problem – is rotational, with

$$f(\theta) \sim -K^2[\tilde{\rho}(\theta)]^{-\frac{1}{2}}, \quad K \equiv (\tfrac{1}{2}c)^{\frac{1}{4}} \gg 1. \tag{4.1}$$

This form for f does not satisfy the no-slip condition on $\theta = \pm\alpha$, and there must again be boundary layers at the walls. We may analyse these boundary layers by first changing to new variables ϕ and η, which we define by posing

$$f(\theta) = K^m P(\theta)\,\phi(\eta), \quad \eta = K^n H(\theta), \tag{4.2}$$

where $H(\theta)$ is to be an $O(1)$ function of θ satisfying $H(-\alpha) = 0$ (for a boundary layer on $\theta = -\alpha$), and the exponents m and n are constants. On substitution of (4.2) into (2.3) it becomes clear that appropriate choices are $m = 2$, $n = 1$ and

$$H(\theta) = \int_{-\alpha}^{\theta} [\tilde{\rho}(\xi)]^{\frac{1}{4}}[\tilde{\mu}(\xi)]^{-\frac{1}{2}}\,d\xi, \quad P(\theta) = [\tilde{\rho}(\theta)]^{-\frac{1}{2}}; \tag{4.3}$$

then the JH equation (2.3) becomes, with no approximation (but provided $c > 0$),

$$\phi'' - 2(1-\phi^2) = -K^{-1}L_1\phi' - K^{-2}L_2\phi, \tag{4.4}$$

where
$$L_1 = \tilde{\rho}^{\frac{1}{2}}\frac{d}{d\theta}\left(\frac{\tilde{\mu}^{\frac{1}{2}}}{\tilde{\rho}^{\frac{3}{4}}}\right), \quad L_2 = \tilde{\mu}\,\frac{d^2P}{d\theta^2} + \frac{d\tilde{\mu}}{d\theta}\frac{dP}{d\theta} + 4\tilde{\mu}P, \quad P = \tilde{\rho}^{-\frac{1}{2}}. \tag{4.5}$$

We may now seek a regular expansion

$$\phi(\eta, K) = \phi_0(\eta) + K^{-1}\phi_1(\eta) + \dots, \tag{4.6}$$

which when substituted into (4.4) leads to the equations

$$\phi_0'' = 2(1-\phi_0^2), \tag{4.7}$$

$$\mathscr{L}(\phi_n) \equiv \phi_n'' + 4\phi_0\phi_n = \Gamma_n(\eta) \quad (n \geqslant 1) \tag{4.8}$$

(cf. (2.16) and (2.17)). By the no-slip condition on $\theta = -\alpha$, the ϕ_n must satisfy

$$\phi_n(0) = 0, \tag{4.9}$$

and, continuing the outer expansion (4.1) in the form

$$f \sim -K^2\tilde{\rho}^{-\frac{1}{2}} - \tfrac{1}{4}L_2\tilde{\rho}^{-\frac{1}{2}}, \quad \phi \sim -1 - \tfrac{1}{4}K^{-2}L_2, \tag{4.10}$$

we see that the appropriate conditions at large η are

$$\phi_0 \to -1, \quad \phi_1 \to 0, \quad \phi_2 \sim -\tfrac{1}{4}L_2 \quad \text{as} \quad \eta \to \infty. \tag{4.11}$$

The function $\phi_0(\eta)$ is clearly identical with the $F_0(\eta)$ in (2.21), so that at leading order

$$f(\theta) \sim K^2[\tilde{\rho}(\theta)]^{-\frac{1}{2}}[2 - 3\tanh^2(\eta+\beta)], \quad \eta = KH(\theta). \tag{4.12}$$

Figure 4 is a sketch of the sort of profile predicted by this solution. For higher orders, the solution of (4.8), (4.9), (4.11) may be written down as in (2.19). At $O(K)$ we have $\Gamma_1 = -L_1F_0'$, which suggests that it would be convenient to replace (4.6) by

$$\phi(\eta, K) = \phi_0(\eta) + K^{-1}L_1\phi_1(\eta) + \dots; \tag{4.13}$$

then ϕ_1 satisfies
$$\mathscr{L}(\phi_1) = -F_0, \quad \phi_1(0) = 0, \quad \phi_1(\infty) = 0, \tag{4.14}$$

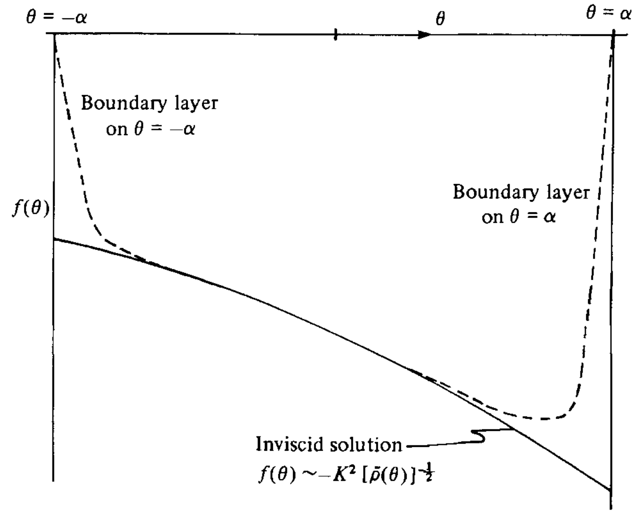

FIGURE 4. Sketch of a velocity profile at large c when there is continuous variation of density (and viscosity). Away from the walls, the profile is the inviscid rotational solution (4.1), and near each wall there is a boundary layer of the form (4.12).

so that

$$\tfrac{10}{3}\phi_1 = 3+2\tanh(\eta+\beta)-5\tanh^2(\eta+\beta)-\tfrac{1}{3}e^{-2(\eta+\beta)} \\ -(5\eta+12-3\surd 6)\tanh(\eta+\beta)\operatorname{sech}^2(\eta+\beta). \quad (4.15)$$

Terms at higher order can similarly be found.

If, exceptionally, $\tilde{\mu}^2/\tilde{\rho}^3$ is constant (with $\tilde{\mu} = \mu_0\tilde{\rho}^{\frac{3}{2}}$, say) then $L_1 \equiv 0$, $\phi_1 \equiv 0$, and

$$\Gamma_2 = -L_2(\theta)\,F_0(\eta), \quad L_2 = -\tfrac{1}{2}\mu_0(\tilde{\rho}''-8\tilde{\rho}). \quad (4.16)$$

In that case the second non-zero term in (4.13) is $O(K^{-2})$:

$$\phi \sim \phi_0(\eta)+\tfrac{1}{4}K^{-2}L_2(\theta)\,F_2(\eta), \quad (4.17)$$

where F_2 is as in (2.22). An even more exceptional case arises if $\tilde{\mu}$ and $\tilde{\rho}$ vary with θ in such a way that

$$L_1(\theta) \equiv 0, \quad L_2(\theta) \equiv 0; \quad (4.18)$$

then the exact JH equation (4.4) becomes

$$\phi'' = 2(1-\phi^2), \quad (4.19)$$

and the expansion (4.6) for large K terminates with ϕ_0 (i.e. (4.12) is valid at all orders, corrections to it being $o(K^N)$ for any N). However, the functions $\tilde{\mu}(\theta)$ and $\tilde{\rho}(\theta)$ that satisfy (4.18) are rather special:

$$\tilde{\rho}(\theta) = \rho_0\cosh 2\surd 2(\theta-\theta_0), \quad \tilde{\mu}(\theta) = \mu_0[\tilde{\rho}(\theta)]^{\frac{3}{2}}. \quad (4.20)$$

Here ρ_0, μ_0 and θ_0 are constants, only one of which is independent, since the normalization conditions (2.1b) require that

$$\rho_0\cosh 2\surd 2\,\theta_0 = \frac{2\surd 2\,\alpha}{\sinh 2\surd 2\,\alpha}, \quad 2\alpha = \mu_0\rho_0^{\frac{3}{2}}\int_{-\alpha}^{\alpha}[\cosh 2\surd 2\,(\theta-\theta_0)]^{\frac{3}{2}}\,d\theta. \quad (4.21)$$

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